ON THE PROBLEM OF A WING OF A GIVEN VOLUME WITH MINIMUM WAVE DRAG

(K ZADACHE O KRYLE ZADANNOGO OB'EMA S MINIMAL'NYM VOLNOVYM SOPROTIVLENIEM

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E. B. BULYGINA
(Novosibirsk)

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The problem is solved on the basis of linearized theory by the method of variation. The solution is obtained for a wing with an arbitrary leading edge, the equation of which is given by the power series.

Let $z(x, y)$ be the equation of the surface of the wing; then the volume of the wing is determined by the double integral of this function, over the area s, which is a projection of the wing upon the plane $z = 0$:

$$v = \iint_s z\,dx\,dy \tag{1}$$

The drag of the wing is determined by a summation of the projections of pressure forces, p, multiplied by the angle of inclination of the surface:

$$C_x = \frac{2}{s}\iint_s p\,\frac{\partial z}{\partial x}\,dx\,dy \qquad \left(\alpha = \frac{\partial z}{\partial x}\right) \tag{2}$$

The pressure at any point $P(s, y)$ on the surface of the wing with a supersonic leading edge is in turn determined by the double integral [1]

$$p = -2\frac{\partial}{\partial x}\iint_\nabla \frac{\alpha(\xi, \eta)}{\sqrt{(x-\xi)^2 - \beta^2(y-\eta)^2}}\,d\xi\,d\eta \tag{3}$$

The area of integration, Δ, represents the portion of the surface of the wing cut out by the forward Mach cone with the apex at point $P(s, y)$.

Thus the problem reduces to an isoparametric determination of the functional (2) for a given value of the functional (1). This problem can be simply solved for the type of surfaces where

$$\frac{\partial^2 \alpha}{\partial y^2} = a_2(x) = \text{const}$$

In that case the wave drag of a wing with a chord b can be determined by the formula suggested by Kogan:

$$C_x = \frac{4}{s\beta} \int_0^b \left\{ \overline{\alpha^2(x)} + \frac{a_2(x)}{\beta^2} \int_0^x \overline{\alpha(\eta)}\,(x-\eta)\,d\eta \right\} dx \tag{4}$$

Changing the order of integration in (4) and introducing the function

$$\varphi(\eta) = \int_\eta^b a_2(x)\,(x-\eta)\,dx \tag{5}$$

Formula (4) can then be written

$$C_x = \frac{4}{s\beta} \int_0^b \left\{ \overline{\alpha^2(x)} + \frac{\overline{\alpha(x)}\,\varphi(x)}{\beta^2} \right\} dx \tag{6}$$

For a given plan form of the wing, function $z = z(x, y)$ must be such that in the plane $z = 0$ it will generate a given wing profile $y = \pm\, \gamma(x)$. This condition will be satisfied if the surface of the wing is represented by

$$z = f(x)\,[\gamma^2(x) - y^2] \tag{7}$$

Then the volume of the wing will be

$$v_0 = \frac{4}{3} \int_0^b f(x)\,\gamma^3(x)\,dx \tag{8}$$

According to the method of Lagrangian multipliers for minimizing functional (6) for the given equation (8) it is necessary to examine Euler's equation for the function $F(x)$:

$$F(x) = \overline{\alpha^2(x)} + \frac{1}{\beta^2}\,\overline{\alpha(x)}\,\varphi(x) + \lambda f(x)\,\gamma^3(x) \tag{9}$$

Considering (5), the Euler equation can be written

$$F_\varphi - \frac{d}{dx}\left\{F_f - \frac{d}{dx}F_{f'}\right\} = 0, \qquad F_\varphi = \frac{\bar{\alpha}}{\beta^2} \tag{10}$$

Noting that F_ϕ can be expressed as a derivative with respect to x;

$$F_\varphi = \frac{1}{\beta^2}\,\frac{d}{dx}\left\{\frac{4}{3} f\gamma^3 + c\right\}$$

The first integral of (10) is

$$-\frac{32}{15}\frac{d}{dx}(f' \cdot \gamma^5) - \frac{16}{3} f\gamma^3\gamma'^2 - \frac{16}{3} f\gamma^4\gamma'' + \frac{8}{\beta^2} \frac{1}{\beta^2} f\gamma^3 = c_1 - \lambda\gamma^3 \tag{11}$$

If the leading edge of the wing passes through the origin ($y = 0$, $x = 0$), the constant C_1 becomes zero. Then, after cancellation by y^2, the first integral (11) reduces to:

$$4f''\gamma^2 + 10f'(\gamma^2)' + 5f\left[(\gamma^2)'' - \frac{1}{\beta^2}\right] = \lambda \tag{12}$$

Since the plan form of the wing is given, function y is known. In the general case, the square of this function can be expressed by the series

$$\gamma^2 = a_0x^2 + a_1x^3 + a_2x^4 + \dots + a_nx^{n+2} + \dots \tag{13}$$

The general solution of (12) is a sum of the particular solution and the general solution of the corresponding homogeneous equation. We seek the latter as a product of a certain power of x, x^ρ and a power series:

$$f_1(x) = x^\rho \sum_{s=0}^{\infty} \alpha_s x^\rho \tag{14}$$

The equation for ρ is

$$F_0(\rho) = 4\rho(\rho - 1) + 10\cdot 2\rho + 5\left(2 - \frac{1}{\beta^2 a_0}\right) = 0$$

Solving it, we obtain two values of ρ

$$\rho_1 = -2 + \sqrt{\frac{3}{2} + \delta}, \qquad \rho_2 = -2 - \sqrt{\frac{3}{2} + \delta} \qquad \left(\delta = \frac{1}{\beta^2 a_0}\right) \tag{15}$$

The coefficients of series (11), corresponding to these values of ρ, are obtained from the infinite system of equations

$$\begin{gathered} a_0\alpha_1F_0(\rho+1) + \alpha_0 a_1F_1(\rho) = 0 \\ a_0\alpha_2F_0(\rho+2) + a_1\alpha_1F_1(\rho+1) + a_2\alpha_0F_2(\rho) = 0 \\ \dots\dots\dots\dots\dots\dots \\ a_0\alpha_nF_0(\rho+n) + a_1\alpha_{n-1}F_1(\rho+n-1) + a_2\alpha_{n-2}F_2(\rho+n-2) + \dots + a_n\alpha_0F_n(\rho) = 0 \\ \dots\dots\dots\dots\dots\dots \end{gathered} \tag{16}$$

The function $F_n(\rho)$ is:

$$F_n(\rho) = 4\rho(\rho - 1) + 10\rho(n+2) + 5(n+2)(n+1) \tag{17}$$

Each equation of the system contains one coefficient more than the preceding, and therefore, by assigning an arbitrary value to α_0, it is possible, in terms of this, to express all the coefficients α_1, ..., α_s.

The particular solution of (12) is determined by the series

$$f_2 = \sum_{k=0}^{\infty} \beta_k x^k \tag{18}$$

coefficients of which are determined by the system of equations analogous to (16):

$$a_0G_0(0)\,\beta_0 = \lambda$$
$$a_0G_0(1)\,\beta_1 + a_1G_1(0)\,\beta_0 = 0$$
$$a_0G_0(2)\,\beta_2 + a_1G_1(1)\,\beta_1 + a_2F_2(0)\,\beta_0 = 0 \tag{19}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_0G_0(k)\,\beta_k + a_1G_1(k-1)\,\beta_k + 1 + \ldots + a_k\widetilde{F}_k(0)\,\beta_0 = 0$$

where

$$G_0(k) = 4k(k-1) + 10\cdot 2k + 5(2-\delta) \tag{20}$$
$$G_n(k) = 4k(k-1) + 10(n+2)k + 5(n+2)(n+1)$$

Since λ is an arbitrary multiplier, β_0 can be considered arbitrary, and all the coefficients of the series (16) can be expressed in terms of it. The function $f(x)$ sought for is determined by the sum of the series:

$$f(x) = \alpha_0(\rho_1)\,x^{\rho_1}\left(1 + \sum_{s=1}^{\infty}\frac{\alpha_s(\rho_1)}{\alpha_0(\rho_1)}x^s\right) +$$
$$+ \alpha_0(\rho_2)\,x^{\rho_2}\left(1 + \sum_{s=0}^{\infty}\frac{\alpha_s(\rho_2)}{\alpha_0(\rho_2)}x^s\right) + \beta_0\left(1 + \sum_{k=1}^{\infty}\frac{\beta_k}{\beta_0}x^k\right) \tag{21}$$

The three arbitrary constants $\alpha_0(\rho_1)$, $\alpha_0(\rho_2)$ and β_0 are determined from the conditions:

1) On the trailing edge of the wing where $x = b$ the function must become zero;

2) The surface of the wing z must have the boundary at $y^2 - y^2(x) \leqslant 0$;

3) The volume of the wing has a given value v_0.

Let us examine a practical case of a delta wing. In this case, series (14) contains only one member. The leading edge is determined by the equation $y = \pm kx$, therefore, in the solution of (21) all the coefficients $\alpha_s(\rho_1) = \alpha_s(\rho_2) = \beta_k = 0$ when $s \geqslant 1$.

According to condition 2, $\alpha_0(\rho_2) = 0$. The arbitrary constant β_0 is obtained by satisfying condition 1:

$$\beta_0 = -\alpha_0(\rho_1)\,b^{\rho_1}$$

For minimum drag, the equation of the surface of a delta wing of a given volume is:

$$z = \alpha_0(\rho_1)\,(x^{\rho_1} - b^{\rho_1})\,(k^2x^2 - y^2) \tag{22}$$

The constant $\alpha_0(\rho_1)$ is determined by the given volume:

$$v_0 = -\frac{1}{3} k^3 \frac{\rho_1}{\rho_1 + 4} b^{\rho_1 + 4} \alpha_0 (\rho_1)$$

The drag of a delta wing, the surface of which is expressed by (7) will be:

$$C_{x_{min}} = \frac{v_0^2}{sb^4} 12\delta \frac{\rho_1 + 4}{\rho_1 + 2} \left[\frac{8\rho_1 + 22}{5} + \delta \right]$$

For a delta wing with supersonic leading edge ($\beta = \delta = 1$), the minimum drag is

$$C_{x_{min}} = 128 v_0^2$$

The drag of the same wing with a wedge profile is $C_x = 180\ v_0^2$.

As seen from the comparison the drag of a delta wing with the found surface (22) is 40 per cent less than that of a delta wing with a wedge profile.

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